Taylor's Series: a demonstration*

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July 24, 2026

The Taylor's series is (also) a fundamental object in mathematics. A function is said to be analytic in a point x_0 if and only if it can be expanded as a Taylor's series about x_0. The Taylor's expansion of a function $f(x)$ about a point x_0 can be demonstrated using integration by parts. Consider the following identity[1] involving a well behaved function $f(\cdot)$:

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(\xi) d\xi. \tag{1}$$

Now we integrate by parts the right side of this expression using $u = f'(\xi) \rightarrow du = f''(\xi)d\xi$ and $dv = d\xi \rightarrow v = \xi$ and we get

$$f(x) - f(x_0) = \xi f'(\xi)|_{x_0}^{x} - \int_{x_0}^{x} \xi f''(\xi) d\xi$$

or

$$f(x) - f(x_0) = x f'(x) - x_0 f'(x_0) - \int_{x_0}^{x} \xi f''(\xi) d\xi.$$

If we subtract and add $x f'(x_0)$ to the right side and collect gives

$$f(x) - f(x_0) = [x f'(x) - x f'(x_0)] + [x f'(x_0) - x_0 f'(x_0)] - \int_{x_0}^{x} \xi f''(\xi) d\xi.$$

$$f(x) - f(x_0) = x\,[f'(x) - f'(x_0)] + (x - x_0) f'(x_0) - \int_{x_0}^{x} \xi f''(\xi) d\xi. \tag{2}$$

We note that $x\,[f'(x) - f'(x_0)]$ can be written as

$$x\,[f'(x) - f'(x_0)] = x \int_{x_0}^{x} f''(\xi) d\xi = \int_{x_0}^{x} x f''(\xi) d\xi. \tag{3}$$

*An Excerpt of the Class Notes for the course PETR571 – Advanced Topics in Drilling – New Mexico Tech – Socorro, NM 87031 – August 31 2003. Prepared and lectured by the author.

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[1]This is called the Fundamental theorem of calculus.

Using these two results we get

$$f(x) - f(x_0) = \int_{x_0}^{x} xf''(\xi) + (x - x_0)f'(x_0) - \int_{x_0}^{x} \xi f''(\xi)d\xi,$$

and rearranging results in

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \int_{x_0}^{x} (x - \xi)f''(\xi)d\xi. \tag{4}$$

This is a very interesting equation. It is not a series (yet), but it expresses $f(x)$ in terms of $f(x_0)$, $f'(x_0)$, and an integral; this expression is exact. We continue, considering now the integration by parts of the integral in Eq.4. For that, we make $u = f''(\xi) \rightarrow du = f'''(\xi)d\xi$ and $dv = (x - \xi)d\xi \rightarrow v = -\frac{(x-\xi)^2}{2}$:

$$\int_{x_0}^{x} (x - \xi)f''(\xi)d\xi = -\frac{(x - \xi)^2}{2} f''(\xi)\Big|_{x_0}^{x} + \int_{x_0}^{x} \frac{(x - \xi)^2}{2} f'''(\xi)d\xi,$$

$$\int_{x_0}^{x} (x - \xi)f''(\xi)d\xi = \frac{(x - x_0)^2}{2} f''(x_0) + \int_{x_0}^{x} \frac{(x - \xi)^2}{2} f'''(\xi)d\xi.$$

Plugging this result into Eq.4 we can see the Taylor's series beginning to take form:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2} f''(x_0) + \int_{x_0}^{x} \frac{(x - \xi)^2}{2} f'''(\xi)d\xi. \tag{5}$$

To obtain a general term, it is appropriate to do the integration by part again:

$$\int_{x_0}^{x} \frac{(x - \xi)^2}{2} f'''(\xi)d\xi = -\frac{(x - \xi)^3}{2 \cdot 3} f'''(\xi)\Big|_{x_0}^{x} + \int_{x_0}^{x} \frac{(x - \xi)^3}{2 \cdot 3} f^{iv}(\xi)d\xi$$

$$\int_{x_0}^{x} \frac{(x - \xi)^2}{2} f'''(\xi)d\xi = \frac{(x - x_0)^3}{3!} f'''(x_0) + \int_{x_0}^{x} \frac{(x - \xi)^3}{3!} f^{iv}(\xi)d\xi$$

This step is important for introducing the factorial in the denominator of the general term. Now we are in position to use mathematical induction, hypothesizing that the n^{th} integral term is

$$\int_{x_0}^{x} \frac{(x - \xi)^{n-1}}{(n-1)!} f^{(n)}(\xi)d\xi.$$

Making $u = f^{(n)}(\xi) \rightarrow du = f^{(n+1)}(\xi)d\xi$ and $dv = \frac{(x-\xi)^{n-1}}{(n-1)!}d\xi \rightarrow v = -\frac{(x-\xi)^n}{n!}$ we get

$$\int_{x_0}^{x} \frac{(x - \xi)^{n-1}}{(n-1)!} f^{(n)}(\xi)d\xi = -\frac{(x - \xi)^n}{n!} f^{(n)}(\xi)\Big|_{x_0}^{x} + \int_{x_0}^{x} \frac{(x - \xi)^n}{n!} f^{(n+1)}(\xi)d\xi$$

$$\int_{x_0}^{x} \frac{(x - \xi)^{n-1}}{(n-1)!} f^{(n)}(\xi)d\xi = \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + \int_{x_0}^{x} \frac{(x - \xi)^n}{n!} f^{(n+1)}(\xi)d\xi$$

Therefore, by starting with Eq.1 and integrating by parts n times we obtain the expression for the Taylor's series:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2}f''(x_0) + \ldots + \frac{(x - x_0)^n}{n!}f^{(n)}(x_0) + R_n(x). \quad (6)$$

where $R_n(x) = \int_{x_0}^{x} \frac{(x-\xi)^n}{n!} f^{(n+1)}(\xi)d\xi$ is the remainder term (called Lagrange remainder).

This is the exact Taylor's expansion of order n of the function $f(x)$ about the point x_0. If we keep integrating by part on and on we get[2]

$$f(x) = f(x_0)+(x-x_0)f'(x_0)+\frac{(x - x_0)^2}{2}f''(x_0)+\ldots+\frac{(x - x_0)^n}{n!}f^{(n)}(x_0)+\ldots = \sum_{n=0}^{\infty}\frac{(x - x_0)^n}{n!}f^{(n)}(x_0)$$

If we truncate the expansion after the $f^{(n)}$ term, we can determine the order of the approximation by examining the remainder.

$$R_n(x) = \int_{x_0}^{x} \frac{(x - \xi)^n}{n!} f^{(n+1)}(\xi)d\xi.$$

Using the first mean value theorem for integrals in the close interval $[x_0, x]$, there exist at least one point c in this interval such that[3]

$$R_n(x) = \int_{x_0}^{x} \frac{(x - \xi)^n}{n!} f^{n+1}(\xi)d\xi = f^n(c)\int_{x_0}^{x} \frac{(x - \xi)^n}{n!}d\xi = f^n(c)\frac{(x - x_0)^{n+1}}{(n+1)!}.$$

Because, $|f^n(c)| \neq \infty$, otherwise $f(x)$ would not be analytic and would have no Taylor's expansion, the truncation error is of the order $o\left[(x - x_0)^{n+1}/(n+1)!\right]$.

The special case in which $x_0 = 0$ is called *Maclaurin Series*.

[2]One should note that $0! = 1$ is NOT *by definition*, or a consequence of the said "empty product". Just consider the definition of factorial, itself: $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 = n \times (n-1)!$. Just use the latter, recursive expression and make $n = 1$. We get immediately $0! = \frac{1}{1!} = 1$.

[3]Under the condition that $f^{(n+1)}$ is continuous, and $(x - \xi)^n/n!$ is integrable and does not change sign in $[x_0, x]$, which is clearly true.